# Lecture 5
# Av. case analysis of quick sort, divide and conquer, mergesort

CS 161 Design and Analysis of Algorithms

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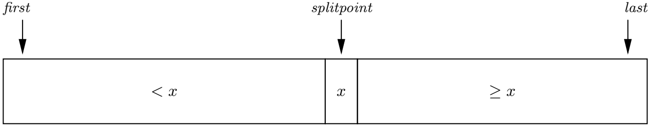
# Pseudocode for Quicksort

```
def quickSort(A,first,last):
    if first < last:
        splitpoint = split(A,first,last)
        quickSort(A,first,splitpoint-1)
        quickSort(A,splitpoint+1,last)
```



# The split step

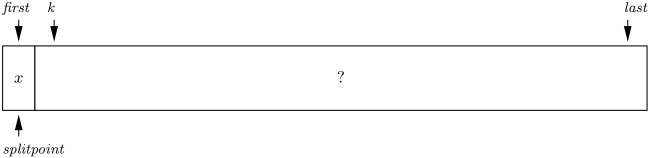
```
def split(A,first,last):
    splitpoint = first
    x = A[first]
    for k = first+1 to last do:
        if A[k] < x:
            A[splitpoint+1] ↔ A[k]
            splitpoint = splitpoint + 1
    A[first] ↔ A[splitpoint]
    return splitpoint
```

Loop invariants:

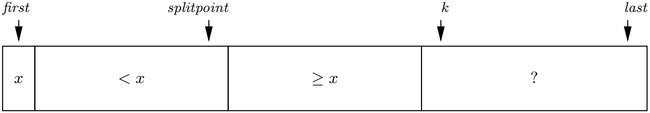
- `A[first+1..splitpoint]` contains keys $< x$.
- `A[splitpoint+1..k-1]` contains keys $\geq x$.
- `A[k..last]` contains unprocessed keys.

# The split step

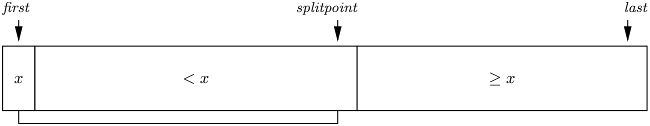
At start:




In middle:




At end:



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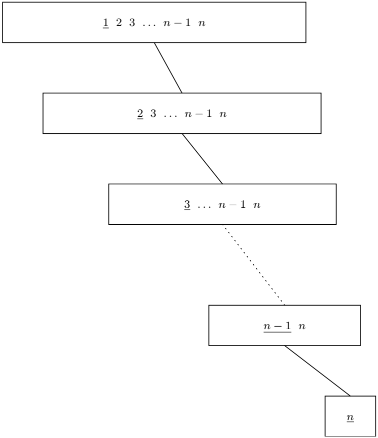


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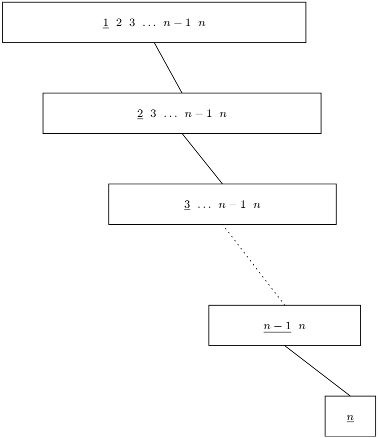


$\binom{n}{2}$ comparisons required. So the worst-case running time for Quicksort is $\Theta(n^2)$. But what about the average case ...?

# Average-case analysis of Quicksort:

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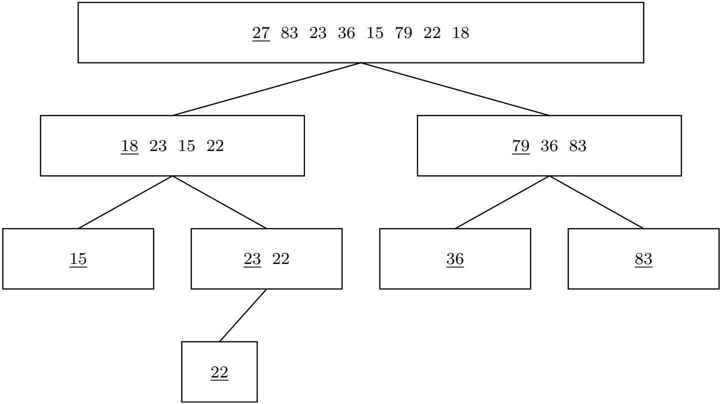
1. Use the binary tree of sorted lists
2. Number the items in sorted order
3. Calculate the probability that two items get compared
4. Use this to compute the expected number of comparisons performed by Quicksort.

# Average-case analysis of Quicksort:




Sorted order: 15 18 22 23 27 36 79 83

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Examples:

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- 23 and 22 (both statements true)

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Examples:

- 23 and 22 (both statements true)
- 36 and 83 (both statements false)

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$$
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P_{i,j} &= \text{The probability that the first key from } \{S_i, S_{i+1}, \ldots, S_j\} \text{ to be chosen as a pivot value is either } S_i \text{ or } S_j \\
&= \frac{2}{j-i+1}
\end{aligned}
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# Average-case analysis of Quicksort

Define indicator random variables $\{X_{i,j} : 1 \leq i < j \leq n\}$

$$X_{i,j} = \begin{cases} 1 & \text{if keys } S_i \text{ and } S_j \text{ get compared} \\ 0 & \text{if keys } S_i \text{ and } S_j \text{ do } \underline{\text{not}} \text{ get compared} \end{cases}$$

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3. The expected value of $X_{i,j}$ is:

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So the average time for Quicksort is $O(n \lg n)$.

# Divide and Conquer

Divide and conquer paradigm

1. Split problem into subproblem(s)
2. Solve each subproblem (usually via recursive call)
3. Combine solution of subproblem(s) into solution of original problem

We will discuss two sorting algorithms based on this paradigm:

- Quicksort (done)
- Mergesort

# MergeSort

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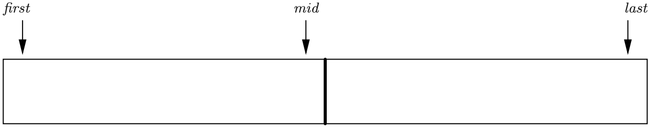
- Split array into two equal subarrays



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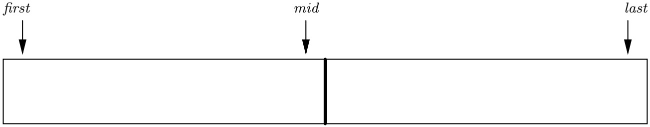

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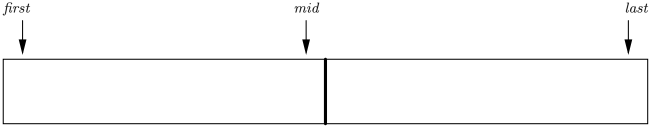

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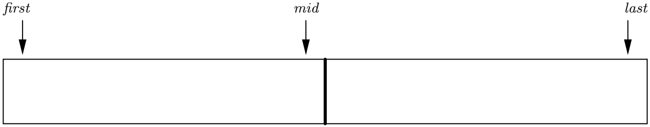


```
def mergeSort(A,first,last):
    if first < last:
        mid = ⌊(first + last)/2⌋
        mergeSort(A,first,mid)
        mergeSort(A,mid+1,last)
        merge(A,first,mid,mid+1,last)
```

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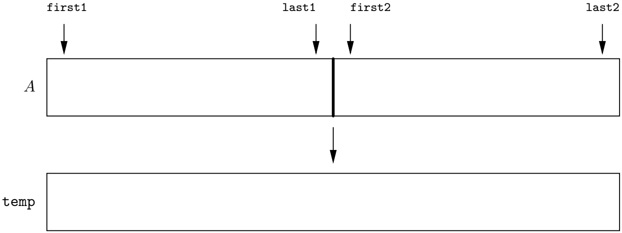

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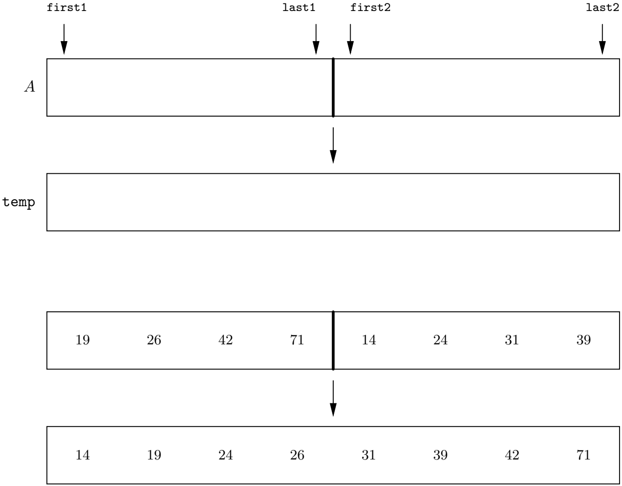

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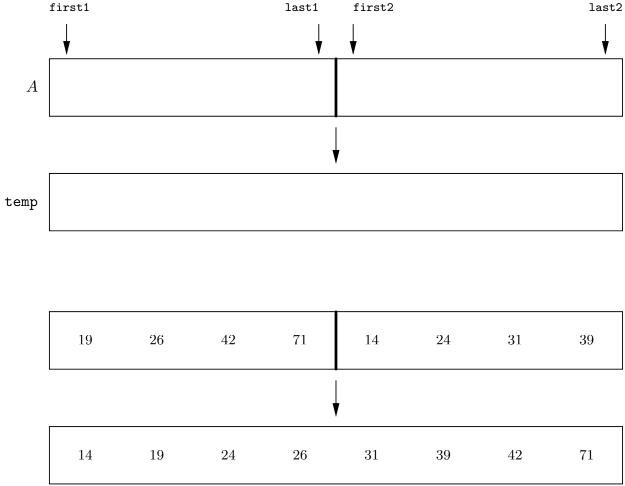


Merging two lists of total size $n$ requires at most $n-1$ comparisons.

# Code for the merge step

```
def merge(A,first1,last1,first2,last2):
    index1 = first1; index2 = first2; tempIndex = 0
   // Merge into temp array until one input array is exhausted
    while (index1 <= last1) and (index2 <= last2)
        if A[index1] <= A[index2]:
            temp[tempIndex++] = A[index1++]
        else:
            temp[tempIndex++] = A[index2++]
   // Copy appropriate trailer portion
    while (index1 <= last1):  temp[tempIndex++] = A[index1++]
    while (index2 <= last2):  temp[tempIndex++] = A[index2++]
   // Copy temp array back to A array
    tempIndex = 0; index = first1
    while (index <= last2):  A[index++] = temp[tempIndex++]
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The exact solution of this recurrence equation is

$$T(n) = n\lceil \lg n \rceil - 2^{\lceil \lg n \rceil} + 1$$

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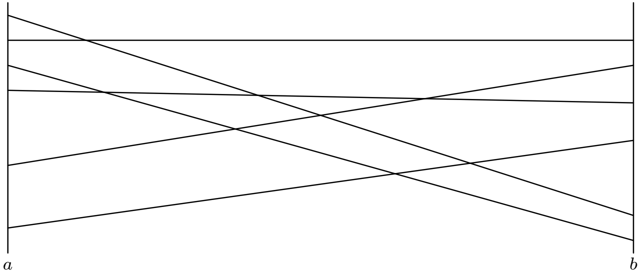
Example: $n = 6$



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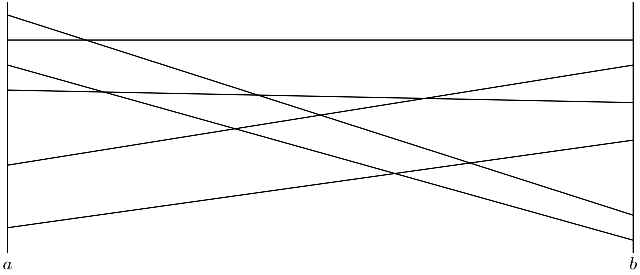

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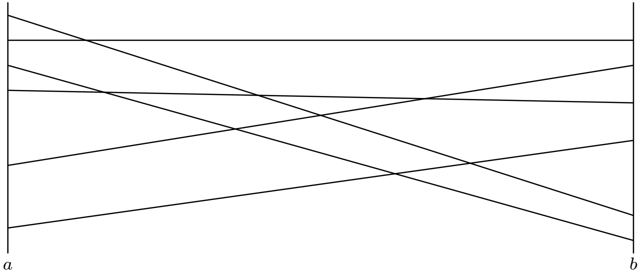

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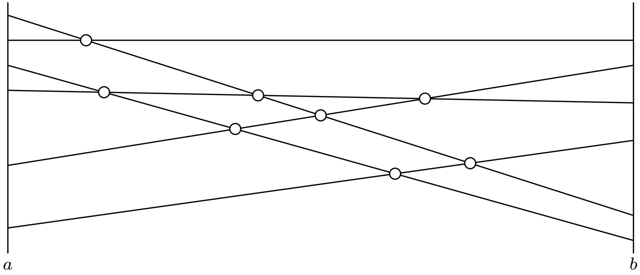

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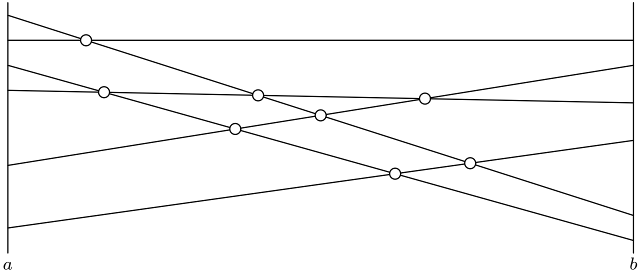


Checking every pair of lines takes $\Theta(n^2)$ time. We can do better.

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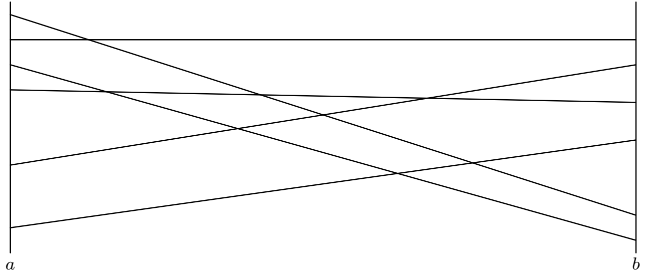

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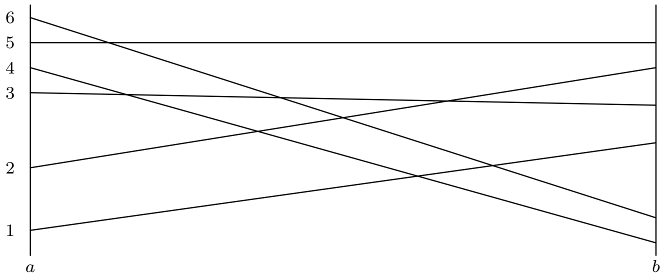

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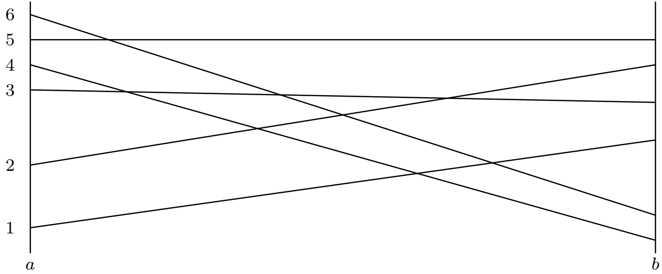

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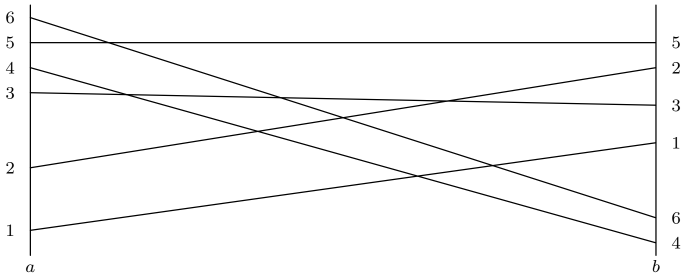

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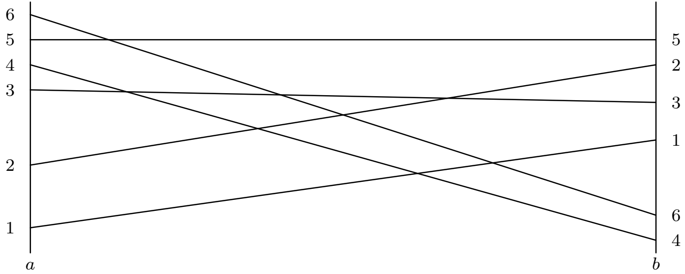

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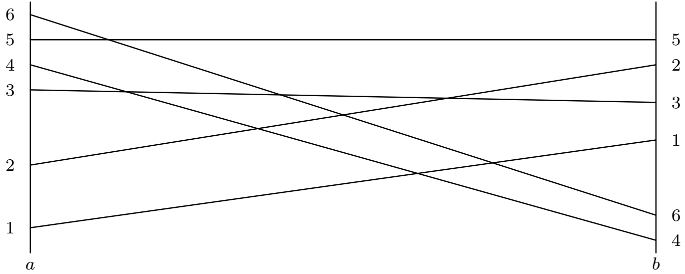

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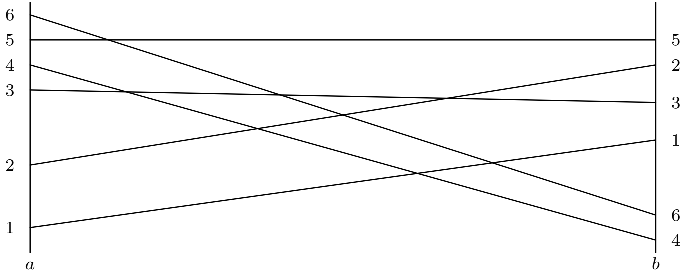


So the problem reduces to counting/reporting inversions.

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In principle, we can use any sorting algorithm to count inversions. Mergesort works particularly nicely.

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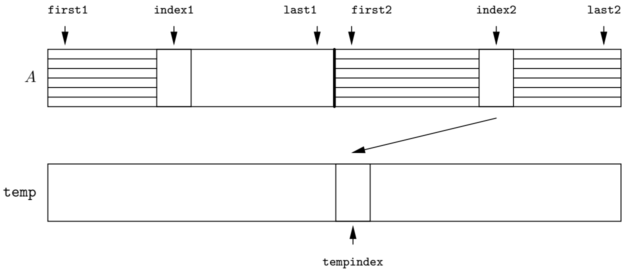

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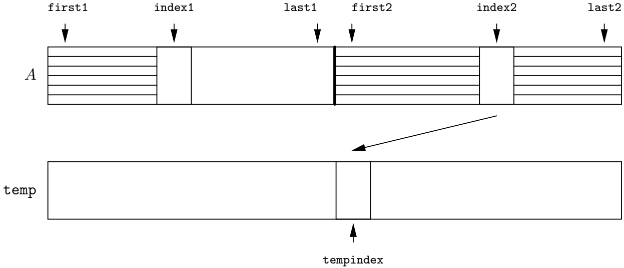


The number of inversions removed is:

$$\texttt{last1} - \texttt{index1} + 1$$

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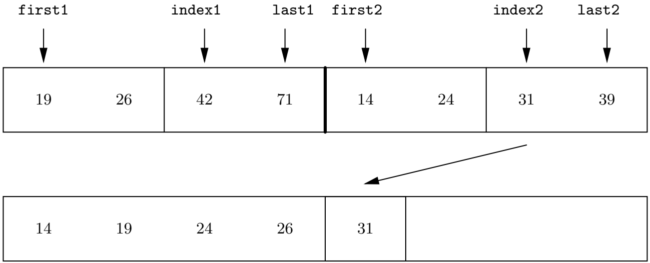
first1
index1
last1
first2
index2
last2
19
26
42
71
14
24
31
39
14
19
24
26
31

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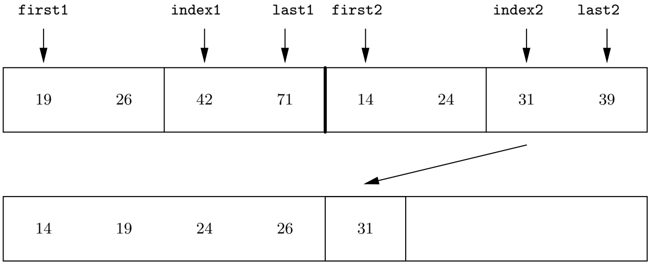


2 inversions removed: $(42, 31)$ and $(71, 31)$

# Pseudocode for the merge step with inversion counting

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```
def merge(A,first1,last1,first2,last2):
    index1 = first1; index2 = first2; tempIndex = 0
    invCount = 0
  // Merge into temp array until one input array is exhausted
    while (index1 <= last1) and (index2 <= last2)
        if A[index1] <= A[index2]:
            temp[tempIndex++] = A[index1++]
        else:
            temp[tempIndex++] = A[index2++]
            invCount += last1 - index1 + 1;
  // Copy appropriate trailer portion
    while (index1 <= last1):  temp[tempIndex++] = A[index1++]
    while (index2 <= last2):  temp[tempIndex++] = A[index2++]
  // Copy temp array back to A array
    tempIndex = 0; index = first1
    while (index <= last2):  A[index++] = temp[tempIndex++]
    return invCount
```

# Pseudocode for MergeSort with inversion counting

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def mergeSort(A,first,last):
    invCount = 0
    if first < last:
        mid = ⌊(first + last)/2⌋
        invCount += mergeSort(A,first,mid)
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Running time is the same as standard mergeSort: $O(n \log n)$

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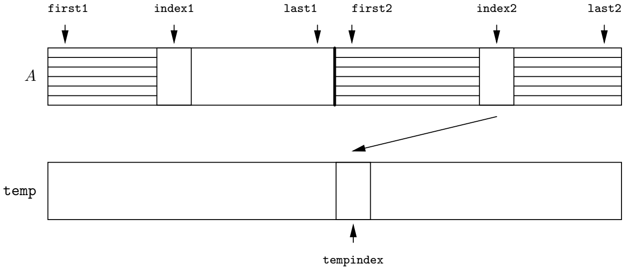


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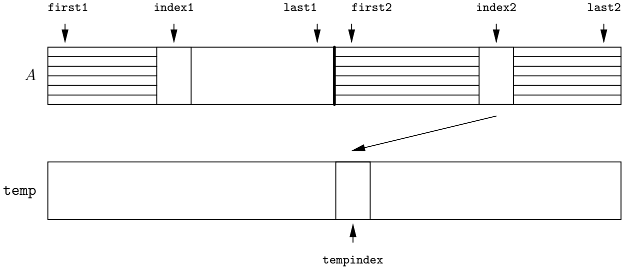


$(A[\texttt{index1}], A[\texttt{index2}])$, $(A[\texttt{index1+1}], A[\texttt{index2}])$, ..., $(A[\texttt{last1}], A[\texttt{index2}])$.

The extra work to do the reporting is proportional to the number of inversions reported.

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The reporting algorithm is an example of an output-sensitive algorithm. The performance of the algorithm depends on the size of the output as well as the size of the input.